COSC 341 – Tutorial 2, Solutions

1. Let $A = \{a, b, c\}$ be a set.

   (a) Define a relation R on A that is irreflexive, asymmetric, and not transitive

   $$R = \{(a, b), (b, c)\}$$

   (b) Extend R to a relation R' that is reflexive

   $$R' = \{(a, b), (b, c), (a, a), (b, b), (c, c)\}$$

   (c) Extend R' to a relation R'' that is symmetric

   $$R'' = \{(a, b), (b, c), (a, a), (b, b), (c, c), (b, a), (c, b)\}$$

   (d) Extend R'' to a relation R''' that is transitive

   $$R''' = \{(a, b), (b, c), (a, a), (b, b), (c, c), (b, a), (c, b), (a, c), (c, a)\}$$

2. Are the following relations reflexive, symmetric, transitive? If they are: How many equivalence classes do they have?

   (a) $\sim$ on $\mathbb{N}$ with: $a \sim b \iff a$ divides b ($\frac{b}{a} \in \mathbb{N}$)

   i. reflexive:

   $$\begin{aligned} &\frac{0}{0} \notin \mathbb{N} \\ &\Rightarrow 0 \nsim 0 \\ &\Rightarrow \sim \text{ is not reflexive} \end{aligned}$$

   ii. symmetric:

   $$\begin{aligned} &\frac{2}{1} \in \mathbb{N} \text{ ,but } \frac{1}{1} \notin \mathbb{N} \\ &\Rightarrow 1 \sim 2, \text{ but } 2 \nsim 1 \\ &\Rightarrow \sim \text{ is not symmetric} \end{aligned}$$

   iii. transitive:

   $$\begin{aligned} &\text{Let } a \sim b,\ b \sim c \\ \Rightarrow\ &\frac{b}{a} \in \mathbb{N},\ \frac{c}{b} \in \mathbb{N} \\ \Rightarrow\ &\frac{c}{a} = \frac{c \cdot b}{a \cdot b} = \frac{c}{b} \cdot \frac{b}{a} \in \mathbb{N}, \\ &\text{because } \frac{c}{b} \in \mathbb{N} \text{ and } \frac{b}{a} \in \mathbb{N} \\ \Rightarrow\ &a \sim c \\ \Rightarrow\ &\sim \text{ is transitive} \end{aligned}$$

   (b) $\sim$ on $\mathcal{P}(\mathbb{N})$ with: $A \sim B \iff A \cap B = \emptyset$

   i. reflexive:

   $$\begin{aligned} &A \cap A = A \neq \emptyset \text{ for all } A \neq \emptyset \\ &\Rightarrow A \nsim A \text{ for all } A \neq \emptyset \\ &\Rightarrow \sim \text{ is not reflexive} \end{aligned}$$

ii. symmetric:

$$\begin{aligned}
&\text{Let } A \sim B \\
\Rightarrow\ & A \cap B = \emptyset \\
\Rightarrow\ & B \cap A = A \cap B = \emptyset \\
\Rightarrow\ & B \sim A \\
\Rightarrow\ & \sim \text{ is symmetric}
\end{aligned}$$

iii. transitive:

$$\begin{aligned}
&\text{Let } A = \{1,2\}, B = \{3,4\}, C = \{1,2\} \\
\Rightarrow\ & A \cap B = \emptyset,\ B \cap C = \emptyset, \text{ but } A \cap C \neq \emptyset \\
\Rightarrow\ & A \sim B,\ B \sim C, \text{ but } A \nsim C \\
\Rightarrow\ & \sim \text{ is not transitive}
\end{aligned}$$

(c) $\sim$ on $\mathbb{N}$ with: $a \sim b \iff a$ and b have the same last digit

i. reflexive:

$$\begin{aligned}
&a \text{ has the same last digit as itself} \\
\Rightarrow\ & a \sim a \\
\Rightarrow\ & \sim \text{ is reflexive}
\end{aligned}$$

ii. symmetric:

$$\begin{aligned}
&\text{Let } a \sim b \\
\Rightarrow\ & a \text{ and } b \text{ have the same last digit} \\
\Rightarrow\ & b \text{ and } a \text{ have the same last digit} \\
\Rightarrow\ & b \sim a \\
\Rightarrow\ & \sim \text{ is symmetric}
\end{aligned}$$

iii. transitive:

$$\begin{aligned}
&\text{Let } a \sim b \text{ and } b \sim c \\
\Rightarrow\ & a \text{ and } b \text{ have the same last digit and } b \text{ and } c \text{ have the same last digit} \\
\Rightarrow\ & a \text{ and } c \text{ have the same last digit} \\
\Rightarrow\ & a \sim c \\
\Rightarrow\ & \sim \text{ is transitive}
\end{aligned}$$

$\Rightarrow \sim$ is an equivalence relation
Each equivalence class is determined by the last digit of one (and hence every) element in it. Thus there are 10 equivalence classes:

$$\begin{aligned}
&\{0, 10, 20, \ldots, 100, 110, 120, \ldots, 1000, 1010, \ldots\} \\
&\{1, 11, 21, \ldots, 101, 111, 121, \ldots, 1001, 1011, \ldots\} \\
&\{2, 12, 22, \ldots, 102, 112, 122, \ldots, 1002, 1012, \ldots\} \\
&\vdots \\
&\{9, 19, 29, \ldots, 109, 119, 129, \ldots, 1009, 1019, \ldots\}
\end{aligned}$$

Homework

1. Are the following relations reflexive, symmetric, transitive? If they are: How many equivalence classes do they have?

   (a) $\sim$ on $\mathcal{P}(\mathbb{N})$ with: $A \sim B \iff A \subseteq B$

      i. reflexive:

$$\begin{aligned} & A \subseteq A \text{ for all } A \in \mathbb{N} \\ \Rightarrow\ & A \sim A \text{ for all } A \in \mathbb{N} \\ \Rightarrow\ & \sim \text{ is reflexive} \end{aligned}$$

      ii. symmetric:

$$\begin{aligned} & \text{Let } A = \{1\}, B = \{1, 2\} \\ \Rightarrow\ & A \subseteq B, \text{ but } B \not\subseteq A \\ \Rightarrow\ & A \sim B, \text{ but} B \nsim A \\ \Rightarrow & \sim \text{ is not symmetric} \end{aligned}$$

      iii. transitive:

$$\begin{aligned} & \text{Let } A \sim B \text{ and } B \sim C \\ \Rightarrow\ & A \subseteq B \text{ and } B \subseteq C \\ \Rightarrow\ & A \subseteq B \subseteq C \\ \Rightarrow\ & A \subseteq C \\ \Rightarrow\ & A \sim C \\ \Rightarrow\ & \sim \text{ is transitive} \end{aligned}$$

   (b) $\sim$ on $\mathbb{Z}$ with: $a \sim b \iff a - b$ is a multiple of 8

      i. reflexive:

$$\begin{aligned} & a - a = 0 = 0 \cdot 8 \text{ is a multiple of 8 for all } a \in \mathbb{N} \\ \Rightarrow\ & a \sim a \text{ for all } a \in \mathbb{N} \\ \Rightarrow\ & \sim \text{ is reflexive} \end{aligned}$$

      ii. symmetric:

$$\begin{aligned} & \text{Let } a \sim b \\ \Rightarrow\ & a - b \text{ is a multiple of 8} \\ \Rightarrow\ & \text{There exists a } k \text{ with } 8k = a - b \\ \Rightarrow\ & b - a = -(a - b) = -8k = 8k' \\ \Rightarrow\ & b - a \text{ is a multiple of 8} \\ \Rightarrow & \sim \text{ is symmetric} \end{aligned}$$

      iii. transitive:

$$\begin{aligned} & \text{Let } a \sim b \text{ and } b \sim c \\ \Rightarrow & \text{There exist } k, l \text{ with } 8k = a - b,\ 8l = b - c \\ \Rightarrow\ & a - c = a - b + b - c = 8k + 8l = 8(k + l) \\ \Rightarrow\ & a \sim c \\ \Rightarrow\ & \sim \text{ is transitive} \end{aligned}$$

$\Rightarrow \sim$ is an equivalence relation
Every natural number is equivalent to a unique number between 0 and 7 inclusive (the remainder on division by 8) so there are 8 equivalence classes.

(c) $\sim$ on $\mathcal{P}(\mathbb{N})$ with: $A \sim B \iff A \cap B \neq \emptyset$

i. reflexive:

$$\begin{aligned} &\text{Let } A = \emptyset \\ \Rightarrow\ &A \cap A = \emptyset \\ \Rightarrow\ &A \nsim A \\ \Rightarrow\ &\sim \text{ is not reflexive} \end{aligned}$$

ii. symmetric:

$$\begin{aligned} &\text{Let } A \sim B \\ \Rightarrow\ &A \cap B \neq \emptyset \\ \Rightarrow\ &B \cap A = A \cap B \neq \emptyset \\ \Rightarrow\ &B \sim A \\ \Rightarrow\ &\sim \text{ is symmetric} \end{aligned}$$

iii. transitive:

$$\begin{aligned} &\text{Let } A = \{1,2\}, B = \{1,3\}, C = \{3,4\} \\ \Rightarrow\ &A \cap B = \{1\} \neq \emptyset,\ B \cap C = \{3\} \neq \emptyset, \text{ but } A \cap C = \emptyset \\ \Rightarrow\ &A \sim B,\ B \sim C, \text{ but } A \nsim C \\ \Rightarrow\ &\sim \text{ is not transitive} \end{aligned}$$