COSC 341 – Tutorial 3, Solutions

1. Show that the set of even natural numbers is countable.

   Let $EN = \{n | n \in \mathbb{N}, n \text{ is even}\}$ denote the set of even natural numbers and let $f : \mathbb{N} \rightarrow EN$ be a function from $\mathbb{N}$ to EN with $f(n) = 2n$. For proving that EN is countable we will prove that f is bijective:

   (a) injectivity:

   $$\begin{aligned}
   &\text{Let } f(n) = f(m) \in EN \\
   &\Rightarrow 2n = f(n) = f(m) = 2m \\
   &\Rightarrow n = m \\
   &\Rightarrow \text{f is injective}
   \end{aligned}$$

   (b) surjectivity:

   $$\begin{aligned}
   &\text{Let } m \in EN \text{ be an arbitrary element of } EN \text{ and let } n = \frac{m}{2} \in \mathbb{N} \\
   &\Rightarrow f(n) = f(\frac{m}{2}) = m \\
   &\Rightarrow \text{f is surjective}
   \end{aligned}$$

2. Show that the set of even integers is countable.

   Let EZ denote the set of even integers and let $f : \mathbb{N} \rightarrow EZ$ with

   $$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd} \end{cases}$$

   For proving that EZ is countable we will prove that f is bijective:

   (a) injectivity:

   $$\begin{aligned}
   &\text{Let } f(n) = f(m) \in EZ \text{ be an arbitrary element of } EZ \\
   &\Rightarrow \text{It is either } f(n) = f(m) \geq 0 \text{ or } f(n) = f(m) < 0 \\
   &\text{If } f(n) = f(m) \geq 0 \\
   &\Rightarrow n = f(n) = f(m) = m \\
   &\Rightarrow n = m \\
   &\text{If } f(n) = f(m) < 0 \\
   &\Rightarrow -n-1 = f(n) = f(m) = -m-1 \\
   &\Rightarrow n = m \\
   &\Rightarrow \text{f is injective}
   \end{aligned}$$

   (b) surjectivity:

   $$\begin{aligned}
   &\text{Let } z \in EZ \\
   &\text{If } z \geq 0 \Rightarrow \text{For } x = z \in \mathbb{N} \text{ it holds } f(x) = f(z) = z \\
   &\text{If } z < 0 \Rightarrow \text{For } x = -z-1 \in \mathbb{N} \text{ it holds } f(x) = f(-z-1) = z \\
   &\Rightarrow \text{f is surjective}
   \end{aligned}$$

3. Show that the set $\{f | f : \mathbb{N} \to \mathbb{N}\}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

   Suppose to the contrary that $\{f | f : \mathbb{N} \to \mathbb{N}\}$ is countable. List each function as $f_0, f_1, ...$

	0	1	2	3	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$\cdots$
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

   Now define the function $f : \mathbb{N} \to \mathbb{N}$ by:

   $$f(n) = f_n(n) + 1$$

   By the definition of f it is different to every function in our list. Therefore our list does not include all possible functions which contradicts our assumption that $\{f | f : \mathbb{N} \to \mathbb{N}\}$ is countable. Therefore $\{f | f : \mathbb{N} \to \mathbb{N}\}$ must be uncountable.

Homework

1. Show that the set of total functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

   As with all diagonal arguments, we argue by contradiction. Suppose, contrary to what we want to prove, that the set of total functions from $\mathbb{N}$ to $\{0, 1\}$ were countable. In that case we could list all these functions as:

   $$f_0, f_1, f_2, \ldots$$

   Now we define a function $g : \mathbb{N} \to \{0, 1\}$ that does not appear in the list (achieving our contradiction). We simply set, for each $i \in \mathbb{N}$:

   $$g(i) = 1 - f_i(i)$$

   (so if $f_i(i) = 1$, $g(i) = 0$ and vice versa). Thus, for any i, g disagrees with f_i at at least one point (namely i) and possibly many others, i.e. $g \neq f_i$ for any i. So our list of functions was *not* complete as we claimed it was, and hence no such list can exist.

2. We can define the set $\mathbb{N}$ of natural numbers as:

   $$0 \in \mathbb{N}$$
   $$\text{If } n \in \mathbb{N}, \text{ then } n + 1 \in \mathbb{N}$$

   We call this a *recursive* definition.
   Give recursive definitions of:

   (a) The set of even natural numbers $EN = \{2n | n \in \mathbb{N}\}$

   $$0 \in EN$$
   $$\text{If } n \in EN, \text{ then } n + 2 \in EN$$

   (b) The set $P = \{1, 2, 4, 8, 16, \ldots\}$ of powers of 2 within $\mathbb{N}$

   $$1 \in P$$
   $$\text{If } n \in P, \text{ then } n + n \in P$$